The Second Spectrum of Timoshenko Beam

Conference Paper · August 2016

DOI: 10.20906/CPS/CON-2016-1050

3 authors:

Ana Carolina Azevedo Vasconcelos
Delft University of Technology

Anderson Soares Da Costa Azevêdo
University of São Paulo

Simone dos Santos
Federal University of Piauí

CONEM 2016
CONGRESSO NACIONAL DE ENGENHARIA MECÂNICA
21 - 25 AGOSTO DE 2016 FORTALEZA - CEARÁ

THE SECOND SPECTRUM OF TIMOSHENKO BEAM

Ana Carolina Azevedo Vasconcelos, carol-2506@hotmail.com[1]
Anderson Soares da Costa Azevêdo , anderson4r2@hotmail.com[1]
Simone dos Santos Hoefel, simone.santos@ufpi.edu.br[1]

[1]Universidade Federal do Piauí − UFPI, Curso de Engenharia Mecânica, Campus Universitário Ministro Petrônio Portella − Bairro Ininga − CEP: 64049-550 − Teresina-PI

Abstract: *Early researchers reported that for hinged-hinged beam, two distinct natural frequencies correspond to the same spatial mode shape and that it establish the existence of a second distinct frequency spectrum. This phenomenon was received the terminology of the "Second Spectrum of Timoshenko Beam". However, some investigators reported that the single frequency spectrum interpretation has the same sets of frequencies calculated from the "two spectra". The objective of this paper is to provide an analysis of the solution of Timoshenko beam equations for higher modes and discuss their complex dynamical behavior. A numerical example presented in literature is re-examined.*
Keywords: *Timoshenko Beam, Second Spectrum, Vibrations*

1. INTRODUCTION

Deflection of beams has interested the human since the century XVII. According to Timoshenko (1953), the first deflection studies were claimed by the Bernoulli family. Jacob Bernoulli (1654-1705) and John Bernoulli (1667-1748) were responsible for beginning of the study of beam deflection and the developing of the principle of virtual displacement, respectively. The John's son, Daniel Bernoulli (1700-1782), continues the study of his uncle and was advisor of Leonhard Euler (1707-1783). With the suggestion gave by his advisor of applying the variational calculus to derive the equation of the deflection curve, Euler formulated in 1744 the Euler-Bernoulli beam also known as the classical theory. Euler-Bernoulli beam theory initially included only prismatic beams. By deducting of Navier (1785-1836) about the neutral axis, there was the generalization to not prismatic beams since that kept the symmetrical cross section. However, the results of the classical theory were not very accurate for the real problems. Saint Venant (1797-1886) noticed that the initial conditions for the beam problem needed to be reformulated, because due increasing thickness, the plane sections do not remain plane and perpendicular.

The next step in the beam theory was given by John William Strutt (1842-1919), known by Lord Rayleigh, which in 1877 tried to resolve the inaccuracy noticed by his advisor, Saint Venant, considering the influence of the rotary inertia. That theory was named of Rayleigh beam. Due to that new formulation occurred a slight improvement in the results for thick beams. Stephen Timoshenko (1878-1972) realized that it was necessary to considerate the shear deformation contribution for the formulation to become closer to reality. Beam theory with both contribution of rotary inertia and shear deformation was initially proposed by Timoshenko (1921). The results of that formulation were more accurate for thicker beams than the theories developed previously by Euler and Bousquet (1744) and Rayleigh (1877). However, this beam model opened space for new mathematical discussions. In 1931, Goens (1931) studied the free-free case and obtained the solution of the differential equation of motion in terms of hyperbolic and sinusoidal terms. In his conclusion, he noticed that from a critical frequency the hyperbolic term became sinusoidal and because of this, changes in the vibration mode were expected. Traill-Nash and Collar (1953) found the governing equation of dynamic motion and the frequency equations for some boundary conditions. In their research, they observed the presence of a new spectre of frequency for free-free and hinged-hinged cases. In 1953 and 1954, Anderson (1953) and Dolph (1954) confirmed the results obtained by Collar.

Second spectrum was discussed during the following decades. Many scientists looked for evidences and tried to understand the physical meaning of this phenomenon. The second spectrum was characterized as a group of frequencies that has the same mode of vibration obtained for lower frequencies. Huang (1961) published the frequency equations and mode of vibration for the classical boundary conditions, but he did not make any comment about the second spectre. In this same year, Vlasov (1961) published his studies about tubular cross section beams of thin wall considering the inertia rotary effect, but without mentions about the second spectre. In 1973, Tobe and Sato (1973) looked for evidences of second spectre in short cantilever beams. In their conclusions, they found difficulty in classifying the frequencies of the two distinct spectra. Thomas and Abbas (1975) were the first to analyse the Timoshenko beam by finite element method. They concluded that the second spectrum only exists for hinged-hinged case. Downs (1976) analysed and concluded that the shear and bending deformation were in phase for modes below the critical frequency, while for modes above this frequency, these deformations were not in phase. This analysis was important for the conclusions of Levison and Cooke later.

In the early 80, Hathout *et al.* (1980) and Bhashyam and Prathap (1981) concluded that the second spectre of frequency exists for any boundary conditions both in damped systems as in undamped. The demonstration of these researchers was questioned by Levinson and Cooke (1982). Levison and Cooke concluded that there was no need of the nomination for the phenomenon of two pairs of natural frequencies above the critical frequency for hinged-hinged beams, because through their analysis, they proved that the pairs of vibration mode only were different for distinct frequencies. In their article, they explained better the physical meaning of this phenomenon. The conclusions of Cooke were confirmed by Han *et al.* (1999) and Stephen (2006). Both observed that this phenomenon also occurs for beams with sliding support ends. In this work it will be presented the formulation of Timoshenko for hinged-hinged and sliding-sliding boundary conditions and one example to discuss the phenomenon of two pairs of natural frequencies that occurring in hinged-hinged beam for values above the critical frequency. The results will be compared to Cooke's work and from them conclusions will be made.

2. FORMULATION OF TIMOSHENKO BEAM THEORY

The coupled differential equations of motion from Timoshenko beam theory are (Timoshenko, 1921):

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} - \frac{EI\rho}{\kappa G}\frac{\partial^4 v}{\partial x^2 \partial t^2} - \rho I\frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{\kappa G}\frac{\partial^4 v}{\partial t^4} = 0, \tag{1}$$

$$EI\frac{\partial^4 \psi}{\partial x^4} + \rho A\frac{\partial^2 \psi}{\partial t^2} - \frac{EI\rho}{\kappa G}\frac{\partial^4 \psi}{\partial x^2 \partial t^2} - \rho I\frac{\partial^4 \psi}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{\kappa G}\frac{\partial^4 \psi}{\partial t^4} = 0, \tag{2}$$

where E is the modulus of elasticity, I is the moment of inertia, ρ is the mass density of the beam, A is the area of the beam section, κ is the shear coefficient, G is the shear modulus, v is the transverse deflection and ψ is the rotation of the beam section. Admitting that the beam is excited harmonically with a frequency f and:

$$v(x,t) = V(x)e^{jft}, \quad \psi(x,t) = \Psi(x)e^{jft} \quad \text{and} \quad \xi = \frac{x}{L}, \tag{3}$$

where $V(x)$ is a normal function of $v(x,t)$, $\Psi(x)$ is a normal function of $\psi(x,t)$, $j = \sqrt{-1}$, ξ is the non-dimensional length of the beam and L is the length of the beam. Substituting the above relations into Eq. (1) through Eq. (2) and omitting the common term e^{jft}, the following equations are obtained (Soares and Hoefel, 2015):

$$\frac{\partial^4 V}{\partial \xi^4} + b^2(r^2+s^2)\frac{\partial^2 V}{\partial \xi^2} - b^2(1-b^2r^2s^2)V = 0, \tag{4}$$

$$\frac{\partial^4 \Psi}{\partial \xi^4} + b^2(r^2+s^2)\frac{\partial^2 \Psi}{\partial \xi^2} - b^2(1-b^2r^2s^2)\Psi = 0, \tag{5}$$

where

$$b^2 = \frac{\rho A L^4}{EI}f^2, \quad r^2 = \frac{I}{AL^2}, \quad s^2 = \frac{EI}{\kappa AGL^2} \quad \text{and} \quad f = 2\pi\omega. \tag{6}$$

Here, r^2 and s^2 are the coefficients associated with the effect of rotatory inertia and shear deformation, respectively, f is the angular frequency and ω is the natural frequency (Wang, 1970). The analytical solution of Eq. (4) and Eq. (5) are given by transcendent functions, given by:

$$V = c\,e^{\lambda\,\xi} \quad \text{and} \quad \Psi = \bar{c}\,e^{\lambda\,\xi}. \tag{7}$$

Substituting the Eq. (7) and its derivatives into Eq. (4), is obtained that:

$$\lambda^4 + b^2\left(r^2+s^2\right)\lambda^2 - b^2(1-b^2\,r^2\,s^2) = 0. \tag{8}$$

Equation (8) has four roots λ, of which two are always imaginary, while the other two are real or imaginary depending on the frequency f. They are real when the frequency f is less than the critical frequency f_c and are imaginary when the frequency f is greater than the critical frequency f_c . This cutoff frequency is given by:

$$f_c^2 = \frac{\kappa GA}{\rho I}. \tag{9}$$

Downs (1976) discussed the case $f^2 = \kappa GA/\rho I$ and identifies the motion for this case as a shear oscillation without transverse deflection. When $f = f_c$, the modal shapes are called shear mode. So, two cases must be considered: when $f < f_c$ and $f > f_c$ (Han *et al.*, 1999). In the first case,

$$\left[(r^2 - s^2)^2 + \frac{4}{b^2}\right]^{\frac{1}{2}} > (r^2 + s^2). \tag{10}$$

Therefore, the solutions of Eq. (4) and Eq. (5) are given by Huang (1961):

$$V = c_1\, cosh(\alpha_1\, b\, \xi) + c_2\, sinh(\alpha_1\, b\, \xi) + c_3\, cos(\beta\, b\, \xi) + c_4\, sin(\beta\, b\, \xi), \tag{11}$$

$$\Psi = \overline{c_1}\, sinh(\alpha_1\, b\, \xi) + \overline{c_2}\, cosh(\alpha_1\, b\, \xi) + \overline{c_3}\, sin(\beta\, b\, \xi) + \overline{c_4}\, cos(\beta\, b\, \xi), \tag{12}$$

where

$$\alpha_1 = \frac{1}{\sqrt{2}}\sqrt{-(r^2 + s^2) + \sqrt{(r^2 - s^2)^2 + \frac{4}{b^2}}} \quad \text{and} \quad \beta = \frac{1}{\sqrt{2}}\sqrt{(r^2 + s^2) + \sqrt{(r^2 - s^2)^2 + \frac{4}{b^2}}}.$$

The constants of Eq. (11) and Eq. (12) are related as follows:

$$\overline{c}_1 = \frac{(\alpha_1^2 + s^2)b}{\alpha_1 L} c_1, \qquad \overline{c}_2 = \frac{(\alpha_1^2 + s^2)b}{\alpha_1 L} c_2, \tag{13}$$

$$\overline{c}_3 = -\frac{(\beta^2 - s^2)b}{\beta L} c_3 \qquad \text{and} \qquad \overline{c}_4 = \frac{(\beta^2 - s^2)b}{\beta L} c_4. \tag{14}$$

Note that the equations Eq. (11) and Eq. (12) depend on trigonometric and hyperbolic functions, due to the nature of the roots defined by the condition Eq. (10). For second case,

$$\left[(r^2 - s^2)^2 + \frac{4}{b^2}\right]^{\frac{1}{2}} < (r^2 + s^2), \tag{15}$$

the solutions of Eq. (4) and Eq. (5) are given respectively by:

$$V = c_1\, cos(\alpha_2\, b\, \xi) + c_2\, sin(\alpha_2\, b\, \xi) + c_3\, cos(\beta\, b\, \xi) + c_4\, sin(\beta\, b\, \xi), \tag{16}$$

$$\Psi = \overline{c_1}\, sin(\alpha_2\, b\, \xi) + \overline{c_2}\, cos(\alpha_2\, b\, \xi) + \overline{c_3}\, sin(\beta\, b\, \xi) + \overline{c_4}\, cos(\beta\, b\, \xi), \tag{17}$$

where

$$\alpha_2 = \frac{1}{\sqrt{2}}\sqrt{(r^2 + s^2) - \sqrt{(r^2 - s^2)^2 + \frac{4}{b^2}}}.$$

The constants of Eq. (16) and Eq. (17) are related as follows:

$$\overline{c}_1 = \frac{(s^2 - \alpha_2^2)b}{\alpha_2 L} c_1, \qquad \overline{c}_2 = \frac{(\alpha_2^2 - s^2)b}{\alpha_2 L} c_2, \tag{18}$$

$$\overline{c}_3 = \frac{(s^2 - \beta^2)b}{\beta L} c_3 \qquad \text{and} \qquad \overline{c}_4 = \frac{(\beta^2 - s^2)b}{\beta L} c_4. \tag{19}$$

This case occurs for higher frequencies ($f > f_c$) and it will be the main topic of this work. Different of the first case, the equations Eq. (16) and Eq. (17) depend only on trigonometric functions, since the condition Eq. (15) has only imaginary roots.

3. FREQUENCY EQUATIONS

Considering a hinged-hinged beam, the boundary conditions are given by:

$$V(0) = 0, \quad \Psi'(0) = 0, \quad V(1) = 0 \quad \text{and} \quad \Psi'(1) = 0. \tag{20}$$

Applying the boundary conditions Eq. (20) and the constants obtained in Eq. (18) and Eq. (19) in the Eq. (16) and Eq. (17), it is possible to determine the frequency equation. For matrix form,

$$\begin{bmatrix} sin(\alpha_2 b) & sin(\beta b) \\ \frac{-(\alpha_2^2 - s^2)b^2}{L}\, sin(\alpha_2 b) & \frac{-(\beta^2 - s^2)b^2}{L}\, sin(\alpha_2 b) \end{bmatrix} \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{21}$$

For a non-trivial solution, the determinant of matrix sines at Eq. (21) must be equal to zero. Therefore, is obtained the frequency equation which is given by (Levinson and Cooke, 1982):

$$sin(\alpha_2\, b)\, sin(\beta\, b) \;=\; 0. \tag{22}$$

Equation (22) is satisfied in two situations: $sin(\,\alpha_2\, b\,) \;=\; 0$ and $sin(\,\beta\, b\,) \;=\; 0$. Solving the Eq. (16) and Eq. (17) for the situation that $sin(\alpha_2 b) = 0$ are obtained that:

$$V = c_2\, sin(\alpha_2\, b\, \xi) \quad \text{and} \quad \Psi = c_2 \frac{(\alpha_2^2 - s^2)b}{\alpha_2 L} cos(\alpha_2\, b\, \xi). \tag{23}$$

Similarly, for the situation that $sin(\beta b) = 0$,

$$V = c_4\, sin(\beta\, b\, \xi) \quad \text{and} \quad \Psi = c_4 \frac{(\beta^2 - s^2)b}{\beta L} cos(\beta\, b\, \xi). \tag{24}$$

Assuming that $(f < f_c)$ the frequency equation for hinged-hinged beam is given by (Huang, 1961):

$$sin(\,\beta\, b\,) \;=\; 0. \tag{25}$$

Considering a sliding-sliding beam, the boundary conditions are given by:

$$\Psi(0) = 0, \quad \kappa GA \left(\Psi(0) - \frac{\partial V(0)}{\partial \xi} \right) = 0, \quad \Psi(1) = 0 \quad \text{and} \quad \kappa GA \left(\Psi(1) - \frac{\partial V(1)}{\partial \xi} \right) = 0 \tag{26}$$

Applying the boundary conditions Eq. (26) and the constants obtained in Eq. (18) and Eq. (19) in the Eq. (16) and Eq. (17), is acquired the matrix

$$\begin{bmatrix} \dfrac{(s^2 - \alpha_2^2)b}{\alpha_2 L} sin(\alpha_2 b) & \dfrac{(s^2 - \beta^2)b}{\beta L} sin(\beta b) \\ \alpha_2 b\, sin(\alpha_2 b) & \beta b\, sin(\beta b) \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{27}$$

Similarly to the hinged-hinged case, is obtained the frequency equation which is given by:

$$\frac{(\beta^2 - \alpha_2^2)s^2}{\alpha_2 \beta} sin(\alpha_2 b) sin(\beta b) = 0. \tag{28}$$

Traill-Nash and Collar (1953) first claimed the existence of double eigenvalue for hinged-hinged boundary conditions for $f > f_c$. They noted that modes shapes given by Eq. (24) are similar to Eq. (23). Due to this phenomena the frequencies found were separated in two distinct spectra: the first spectrum which is represented by $sin(\beta b) = 0$ and the second spectrum which is represented by $sin(\alpha_2 b) = 0$. However, Downs (1976) noted that mode shapes of each spectra of frequencies although similar are distinct, because deformations due to shear and deflection are of the same phase and are summed to give the total transverse deflection for $f < f_c$, while for $f > f_c$ the shear and bending deformation are opposed with the net transverse deflection equal to their difference as can be observed in Fig. 1.

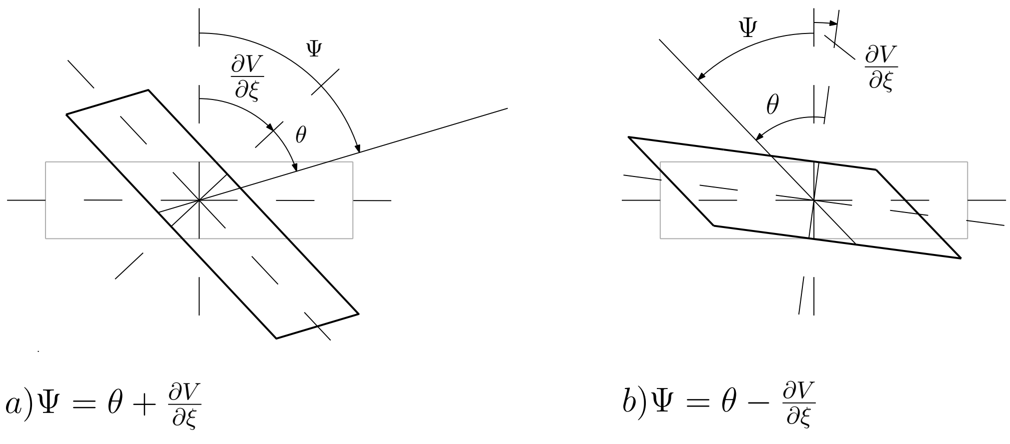

Figure 1: Free vibration of Timoshenko beam: (a) $f < f_c$; (b) $f > f_c$.

4. NUMERICAL EXAMPLE

Consider a hinged-hinged beam of rectangular cross section (Fig. 2) such that $L = 0.5\,m$, $H/L = 0.25$, $k = 5/6$, $E = 210\,GPa$, $G = 80.8\,GPa$ and $\rho = 7850\,kg/m^3$. Replacing theses values at inertia rotary and shear deformation factors, it has $r = 0.0722$ and $s = 0.1275$.

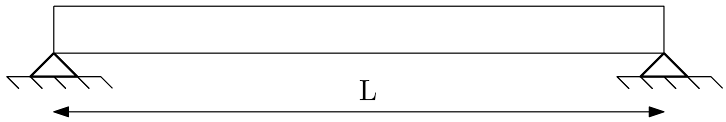

Figure 2: Hinged-hinged beam.

This example was presented by Levison and Cooke in 1982. Their results can be checked on the Tab.1, while the values obtained in this work are on the Tab.2. Both Tab.(1-2) provides the first 15 natural frequencies of this beam: the first ten frequencies from the first spectrum, the first five frequencies from the second spectrum, and the shear mode frequency. The calculus of the frequency equation roots was used False Position method.

Table 1: Natural frequencies (rad/s) obtained by Levison and Cooke.

1^{st} spectrum	Frequency	critical frequency	Frequency	2^{nd} spectrum
1	6712			
2	22136			
3	40701			
4	60170			
Shear Mode		65851		
5	79806			
			89077	1
6	99375			
			108044	2
7	118812			
			132212	3
8	138112			
9	157287			
			158992	4
10	176356			

Table 2: Natural frequencies (rad/s).

1^{st} spectrum	Frequency	critical frequency	Frequency	2^{nd} spectrum
1	6712			
2	22137			
3	40705			
4	60177			
5	79816			
Shear Mode		81164		
			89091	1
6	99389			
			108057	2
7	118830			
			132224	3
8	138134			
9	157313			
			159003	4
10	176385			
			187311	5

Analyzing the Tab.1 and Tab.2, it can be noticed two disagreements: the difference between the shear mode values and the position of the fifth natural frequency ω_5 for the first spectrum. A typographic error appear to be present in Levinson and Cooke (1982), for the shear mode value, which do not appear to have been noted in the literature.

Figure 3 through 5 are shown the firsts four modal shapes for hinged-hinged beam. Observe that the frequencies obtained by Eq. (23) and Eq. (24) can be ordered and are not repeated, although these two frequencies are very close together. In the Fig. 3 are presented the four modal shapes for the "first spectrum" and for the "second spectrum". The first plot of Fig. 3 are given by Eq. (24) for the first four frequency values below critical frequency ($f < f_c$) in Tab.

2, while the second plot are given by Eq. (23) for the frequency values above critical frequency ($f > f_c$) for "second spectrum". Notice that although wave forms presented are similar, they are distinct because deformations due to shear and deflection are of the same phase when $f < f_c$ and are out of phase when $f > f_c$ as presented in Fig. 1.

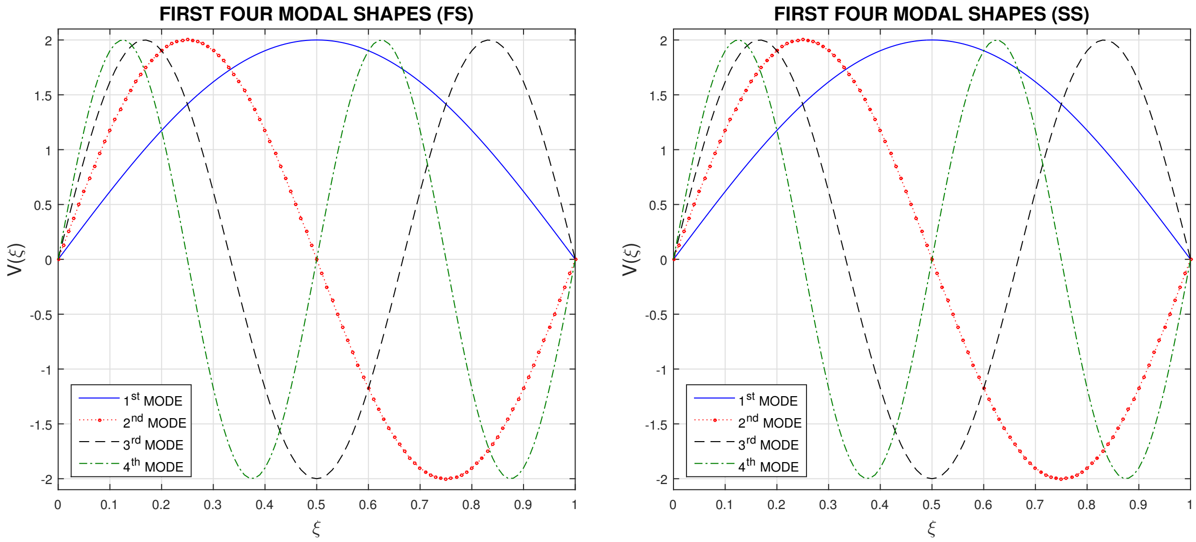

Figure 3: Modal shapes for frequencies below critical frequency.

Figure 4 and 5 presents the first four modal shapes for $f > f_c$ on "first spectrum". Mode shapes are obtained for frequencies presented in Tab. 2. Note that the first mode shape occurs for f_7, the second for f_9, the third for f_{11} and fourth to f_{12} (these indexes correspond to the frequencies for each modal shape).

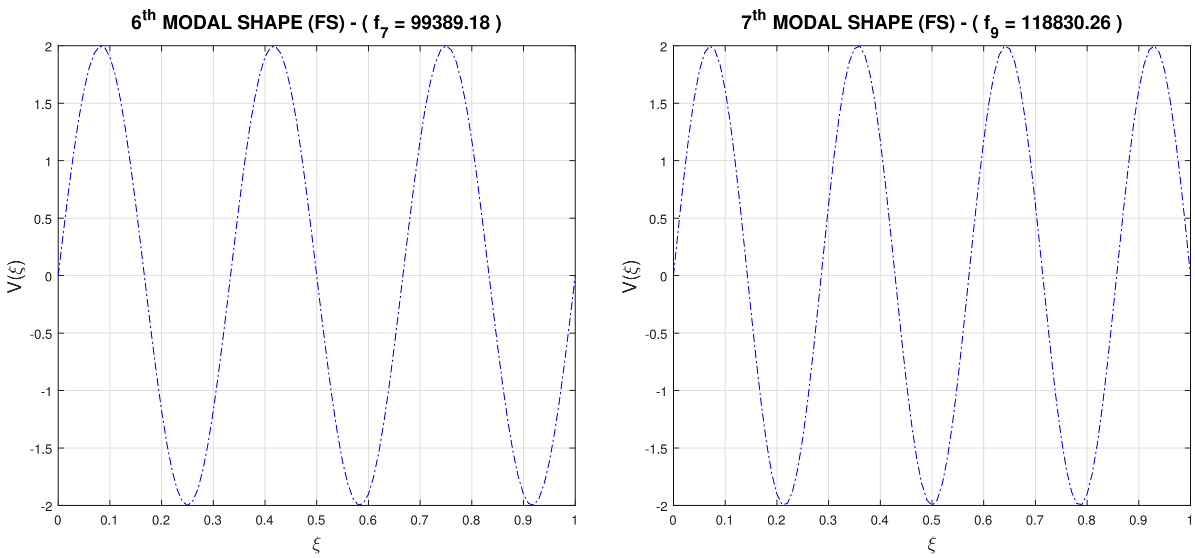

Figure 4: Modal shapes for frequencies above critical frequency.

Figure 6 presents the mode shapes for Eq. (23) and Eq. (24) when $f = f_c$. Levinson and Cooke (1982) observed that shear mode plot was misunderstood by early researchers as we can see in the first plot on Fig. 6. In fact, particulars normalizations mask the phenomena and give the casual impression that the shear mode frequency does provide such a boundary between lower and higher frequencies.

5. CONCLUSIONS

This article discusses the existence of a phenomenon known as the second spectrum. A brief literature review was presented and frequency equation governing the free vibration of Timoshenko beams is then derived for sliding-sliding and hinged-hinged boundary conditions. A numerical example of a hinged-hinged beam of rectangular cross-section considered by Levinson and Cooke was re-examined. Is convenient, therefore, to use the terminology of a single frequency spectrum for the simply supported Timoshenko beam since the eigenvalues can be computed in serial order. However, for the computation and the relevant literature the "second spectrum" terminology can be maintained.

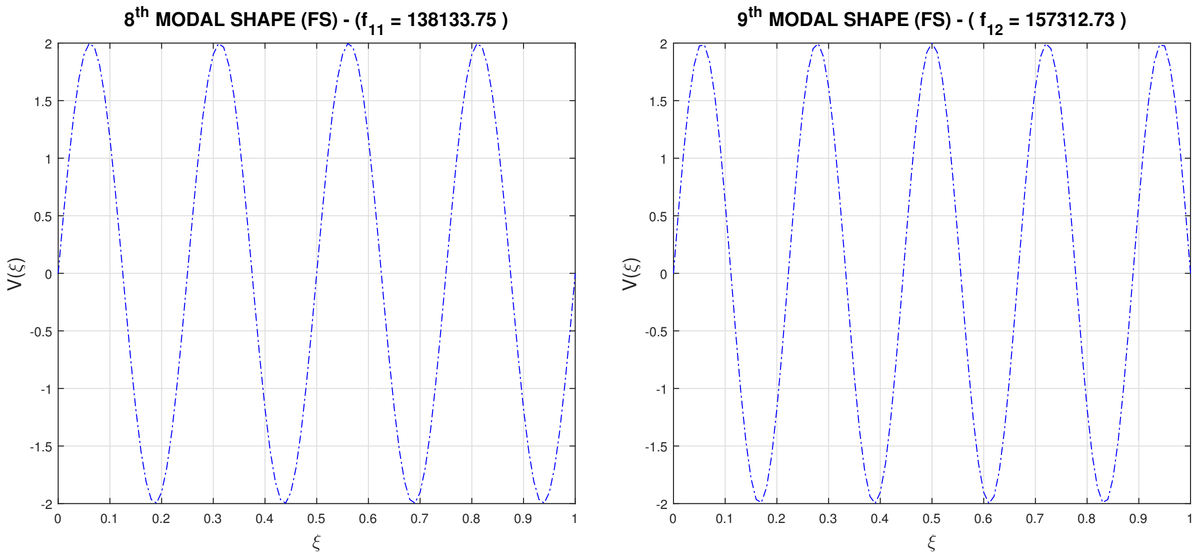

Figure 5: Modal shapes for frequencies above critical frequency.

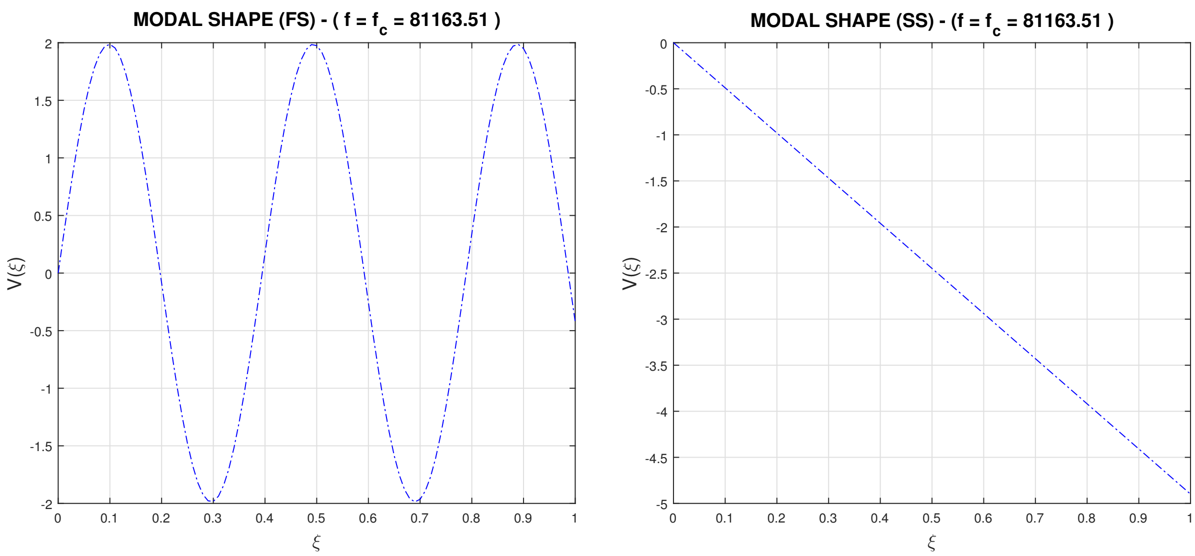

Figure 6: Modal shapes for shear mode.

6. REFERENCES

Anderson, R.A., 1953. "Flexural vibration in uniform beams according to the timoshenko theory". *Journal of Applied Mechanics*, Vol. 20, pp. 504–510.

Bhashyam, G.R. and Prathap, G., 1981. "The second frequency spectrum of timoshenko beam". *Journal of Sound and Vibration*, Vol. 76, pp. 407–420.

Dolph, C., 1954. "On the timoshenko theory of transverse beam vibrations". *Quarterly of Applied Mathematics*, Vol. 12, pp. 175–187.

Downs, B., 1976. "Vibration of a uniform, simply supported timoshenko beam without transverse deflection". *Journal of Applied Mechanics*, Vol. 43, pp. 671–674.

Euler, L. and Bousquet, M.M..C., 1744. *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive Solutio problematis isoperimetrici latissimo sensu accepti. Auctore Leonhardo Eulero ..* apud Marcum-Michaelem Bousquet & Socios.

Goens, E., 1931. "Uber die bestimmung des elastizitätsmodulus von stäben mit hilfe von biegungsschwingungen". *Annalen der Physik*, Vol. 11, pp. 649–678.

Han, S.M., Benaroya, H. and Wei, T., 1999. "Dynamics of transversely vibrating beams using four engineering theories". *Journal of Sound and Vibration*, Vol. 225 (5), pp. 935–988.

Hathout, I., Leipholz, H. and Singhal, K., 1980. "Sensitivity of the frequencies of damped timoshenko beams". *Journal of Engineering Sciences*, Vol. 6, pp. 113–121.

Huang, T.C., 1961. "The effect of rotatory inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions". *Journal of Applied Mechanics*, Vol. 28, pp. 579–584.

Levinson, M. and Cooke, D.W., 1982. "On the two frequency spectra of timoshenko beams". *Journal of Sound and*

Vibration, Vol. 84, pp. 319–326.
Rayleigh, J., 1877. *The Theory of Sound*. Macmillan, New York, 2nd edition.
Soares, A. and Hoefel, S., 2015. "Modal analysis for free vibration of four beam theories". In *Proceedings of the 23rd International Congress of Mechanical Engineering - COBEM2015*. Rio de Janeiro, Brazil.
Stephen, N.G., 2006. "The second spectrum of timoshenko beam theory - futher assessment". *Journal of Sound and Vibration*, Vol. 292, pp. 372–389.
Thomas, J. and Abbas, B.A.H., 1975. "Finite element model dor dynamic analysis of timoshenko beam". *Journal of Sound and Vibration*, Vol. 41, pp. 291–299.
Timoshenko, S.P., 1921. "On the correction for shear of the differential equation for transverse vibration of prismatic bars". *Philosophical Magazine*, Vol. 41, pp. 744–746.
Timoshenko, S.P., 1953. *History of strength of materials*. McGrawl-Hill, United States of America, 1st edition.
Tobe, T. and Sato, K., 1973. "Lateral vibrations of cantilevers considering the effects of rotary inertia, shear flexibility and solid viscosity". *Bulletin of the Japanese Society of Mechanical Engineers*, Vol. 16, pp. 696–703.
Traill-Nash, R.W. and Collar, A.R., 1953. "The effects of shear flexibility and rotatory inertia on bending vibrations beams". *Quaterly Journal of Mechanics and Applied Mathematics*, Vol. 6, pp. 186–213.
Vlasov, V.Z., 1961. *Thin-walled elastic beams*. Jerusalem : published for the National Science Foundation, Washington, D.C., by the Israel Program for Scientific Translations : [available from the Office of Technical Services, U.S. Dept of Commerce, Washington], 2nd edition.
Wang, T.M., 1970. "Natural frequencies of continuos timoshenko beams". *Journal of Sound and Vibrations*, Vol. 13 (4), pp. 409–414.

7. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.