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DYNAMIC BEHAVIOUR OF AN AXIAL-LOADED TIMOSHENKO BEAM ON THE ELASTIC FOUNDATION AND SECOND SPECTRUM ANALYSIS

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Abstract: *The dynamic behavior of systems subject to soil-structure interaction is of great importance in civil engineering. Among the applications of this multiphysics problem can be highlighted railroad tracks, highway pavement, continuously supported pipelines and strip foundations. In this context, the aim of this paper is to use the vibration analysis by finite element method to study the behavior and the second spectrum of uniform beams supported on two-parameter elastic foundation. The first foundation parameter is modeled using linear springs similar to the Winkler foundation and the second foundation parameter is modeled as a shear layer as a function of the total slope of the beam. The effects of axial force, foundation stiffness parameters, transverse shear deformation and rotatory inertia are incorporated into the accurate vibration analysis. The motion equation is derived using Hamilton's variational principle based on the finite element Method (FEM). A finite element is developed by means of the Rayleigh-Ritz using a cubic and quadratic approximated polynomial. The finite element and the analytic solutions are compared to verify the accuracy of the method in this kind of problem. The results obtained are discussed and compared with results obtained by other researchers. The rotary inertia parameter, the axial load parameter and the foundation stiffness influence in the frequency parameter are investigated. The results showed that the axial load decrease the frequency parameter and the foundation increase the frequency parameter. The second spectrum was studied for the hinged-hinged beam presenting a tiny difference for the addition of the foundation and the axial load. In addition, the second spectrum was not verified for the clamped-clamped beam, as expected. Finally, the finite element results present well agreement with the other researches results.*
keywords: *Timoshenko Beam Theory, Elastic Foundation, Finite Element Method, Second Spectrum.*

1. INTRODUCTION

The study of the beams with different dispositions have great importance for many applications in the engineering. In this context, the analysis of soil-structure interaction stands out, where the foundation are coupled to the classic theory of the beams such as Euler-Bernoulli and Timoshenko (1921). Applications of soil-structure problems involve railroad tracks, highway pavements, continuously supported pipelines and strip foundations (Yokoyama, 1996).

On the other hand, the analysis of soil-structure interaction presents a great mathematical and physical complexity due to the great variety and conditions of soil encountered in engineering practice, which makes it difficult to model the soil in a generalized way (Selvadurai, 2013). Thus, many idealized model have been was developed over the years to facilitate the analysis of soil-foundation interaction.

The model proposed by Winkler (1867) is the basis of soil-foundation interaction studies, also known as one-parameter model, where the soil is modeled by elastics linear springs and it is assumed that the soil displacement at the surface is directly proportional to the stress applied at the point. However, the problem associated with this approach is that the displacement only occurs where the load is applied, e.g., outside of the loaded region the displacement is zero. Thus, several researchers have proposed the two-parameter models to correct the shortcomings of the Winkler model. Filonenko-Borodich (1940) was who proposed the first two-parameter model, followed by several researchers such as (Hetenyi, 1979; Pasternak, 1954; Kerr, 1965; Reissner, 1958; Vlasov and Leontiev, 1966). Two-parameter models are defined by two independent elastic constants, where there is a coefficient representing the Winkler parameter and another coefficient of stiffness that varies according to the model.

The Pasternak and Vlasov models are the most used in studies for beam-foundation analysis, because they present great accurate. The Pasternak model is an extension of the Winkler model and presents a Winkler parameter and another parameter that represents the stiffness of a shear layer that is deformed by shear only. However the Vlasov model is the most refined and similarly to the Pasternak model has a Winkler parameter, but the second parameter is proposed a linear or exponential variation with the soil depth (Dutta and Roy, 2002; Auciello, 2008).

Several researchers over the years have sought describe and analyze problems of soil-structure interaction as well. Doyle and Pavlovic (1982), studied the effect of the elastic foundation in the natural frequencies of the beam and columns. Eisenberger and Clastornik (1987) analyzed the vibration and buckling of the beams resting on elastic foundation with two parameters variable. Cheng and Pantelides (1988) used the stiffness dynamic approach to analyze the free vibration of a Timoshenko beam-column in a elastic media. Arboleda-Monsalve *et al.* (2008) used the finite element method (FEM) to analyzed the dynamic of a Timoshenko beam-column resting on elastic foundation with general constraints ends, Kien (2008) studied through of the FEM the vibration of a prestressed Timoshenko beam partially or fully supported above a elastic foundation and Abohadima *et al.* (2015) investigated the free and forced vibration of an axial-loaded Timoshenko beam resting on a elastic foundation using the Recursive Differentiation Method (RDM). Soares *et al.* (2017), also using the FEM, investigated the dynamic behaviour of a Timoshenko beam resting on a Pasternak foundation.

From this, the objective of this work is to investigate the dynamic behavior of an axial-loaded Timoshenko beam resting on a two-parameter elastic foundation. The FEM is developed and validated by comparing with the analytical solution. Foundation parameters and the axial load contributions are investigated in several numerical examples for beams with different end constraints. Futhermore, the frequency curves are plotted and the phenomena of second spectrum was verified for hinged-hinged end boundary condition.

2. AXIAL-LOADED BEAM ON ELASTIC FOUNDATION FORMULATION

Differential Equations, Eqs. (1) and (2), express the motion for a free vibrating beam, according to Timoshenko's formulation taking into account the foundation and the axial load, as show in Fig. 1 (Wang and Stephens, 1977; Wu, 2013). Where L is the length of the beam, f the axial load, k_w the elastic linear spring parameter and G_p the shear layer parameter.

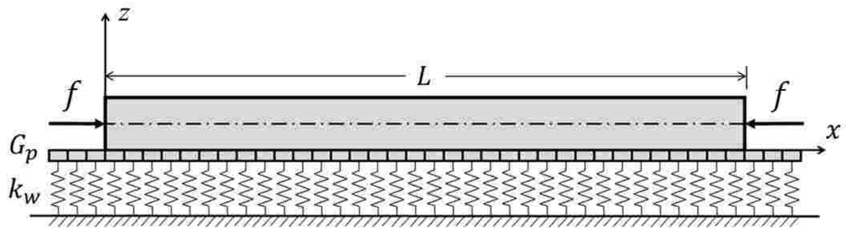

Figure 1: Sketch of an axial-loaded beam resting on a Pasternak foundation.

$$EI\left(1+\frac{G_p-f}{\kappa GA}\right)\frac{\partial^4 w(x,t)}{\partial x^4}+\left(\rho A+\frac{\rho I k_w}{\kappa GA}\right)\frac{\partial^2 w(x,t)}{\partial t^2}-\left(\frac{EIk_w}{\kappa GA}+G_p-f\right)\frac{\partial^2 w(x,t)}{\partial x^2}-$$
$$\rho I\left(1+\frac{E}{\kappa G}+\frac{G_p-f}{\kappa GA}\right)\frac{\partial^4 w(x,t)}{\partial x^2\partial t^2}+\frac{\rho^2 I}{\kappa G}\frac{\partial^4 w(x,t)}{\partial t^4}+k_w w(x,t)=0, \tag{1}$$

$$EI\left(1+\frac{G_p-f}{\kappa GA}\right)\frac{\partial^4 \psi(x,t)}{\partial x^4}+\left(\rho A+\frac{\rho I k_w}{\kappa GA}\right)\frac{\partial^2 \psi(x,t)}{\partial t^2}-\left(\frac{EIk_w}{\kappa GA}+G_p-f\right)\frac{\partial^2 \psi(x,t)}{\partial x^2}-$$
$$\rho I\left(1+\frac{E}{\kappa G}+\frac{G_p-f}{\kappa GA}\right)\frac{\partial^4 \psi(x,t)}{\partial x^2\partial t^2}+\frac{\rho^2 I}{\kappa G}\frac{\partial^4 \psi(x,t)}{\partial t^4}+k_w \psi(x,t)=0. \tag{2}$$

where, E is the Young's modulus, I the moment of inertia, ρ the mass density, A the cross-sectional area, t the time, κ the shear factor, G the shear modulus, $w(x,t)$ is vertical displacement and $\psi(x,t)$ is total slope.

Assuming that the beam is excited harmonically with a frequency ω_n and

$$w(x,t)=W(x)e^{i\omega_n t},\ \psi(x,t)=\Psi(x)e^{i\omega_n t} \text{ and } \xi=x/L, \tag{3}$$

where $W(x)$ and $\Psi(x)$ are the amplitudes of $w(x,t)$ and $\psi(x,t)$, respectively, $i=\sqrt{-1}$, ω_n is the natural frequency, ξ is the dimensionless length and L the length of the beam.

Substituting Eq. (3) into the Eqs. (1) and (2) and omitting the common factor $e^{i\omega_n t}$, the following equations are obtained:

$$\frac{EI}{L^4}\left(1+\frac{G_p-f}{\kappa GA}\right)\frac{d^4W(\xi)}{d\xi^4}+\left[\frac{\rho I}{L^2}\left(1+\frac{E}{\kappa G}+\frac{G_p-f}{\kappa GA}\right)\omega_n^2-\frac{1}{L^2}\left(\frac{EIk_w}{\kappa GA}+G_p-f\right)\right]\frac{d^2W(\xi)}{d\xi^2}+$$
$$\left[\frac{\rho^2 I}{\kappa G}\omega_n^4+k_w-\left(\rho A+\frac{\rho I k_w}{\kappa GA}\right)\omega_n^2\right]W(\xi)=0, \tag{4}$$

$$\frac{EI}{L^4}\left(1+\frac{G_p-f}{\kappa GA}\right)\frac{d^4\Psi(\xi)}{d\xi^4}+\left[\frac{\rho I}{L^2}\left(1+\frac{E}{\kappa G}+\frac{G_p-f}{\kappa GA}\right)\omega_n^2-\frac{1}{L^2}\left(\frac{EIk_w}{\kappa GA}+G_p-f\right)\right]\frac{d^2\Psi(\xi)}{d\xi^2}+$$

$$\left[\frac{\rho^2 I}{\kappa G}\omega_n^4 + k_w - \left(\rho A + \frac{\rho I k_w}{\kappa GA}\right)\omega_n^2\right]\Psi(\xi) = 0, \tag{5}$$

Introducing dimensionless parameters in the Eqs. (4) and (5):

$$b^2 = \frac{\rho AL^4}{EI}\omega_n^2, \quad e^2 = \frac{k_w L^4}{EI}, \quad n^2 = \frac{fL^2}{EI}, \quad p^2 = \frac{G_p L^2}{EI}, \quad r^2 = \frac{I}{AL^2} \quad \text{and} \quad s^2 = \frac{EI}{\kappa GAL^2}, \tag{6}$$

where b is the frequency parameter, e is the linear elastic springs parameter, n is the axial load parameter, p is the shear layer parameter, r is the rotational inertia parameter and s is the shear parameter. So we can write:

$$W^{''''} + \eta_1 W^{''} + \eta_2 W = 0 \quad \text{and} \quad \Psi^{''''} + \eta_1 \Psi^{''} + \eta_2 \Psi = 0, \tag{7}$$

where

$$\eta_1 = \frac{b^2(r^2+s^2) + b^2 s^2 r^2 (p^2 - n^2) - e^2 s^2 - p^2 + n^2}{1 + s^2(p^2 - n^2)} \quad \text{and} \quad \eta_2 = \frac{b^4 s^2 r^2 - b^2(1 + e^2 s^2 r^2) + e^2}{1 + s^2(p^2 - n^2)}. \tag{8}$$

Equation (8) shows that the beam foundation theory with an axial load represents a generalization of the beam theory. Disregarding the parameters n, e and p, the solution returns to the solution of a beam without foundation. Furthermore, Pasternak's foundation theory is a higher generalization, as it includes the solution for the Winkler foundation when $p = 0$.

Assuming the solution of the type $W(\xi) = Ce^{z\xi}$ and $\Psi(\xi) = \overline{C}e^{z\xi}$ and replacing in the Eq. (7), we can obtain the following characteristic equation:

$$z^4 + \eta_1 z^2 + \eta_2 = 0, \tag{9}$$

To solve the O.D.E. of Eq. (7), two conditions must be considered. In the first case, suppose:

$$\eta_1 < 0, \text{ which leads to: } \frac{1}{rs} > b > e \text{ or } \frac{1}{rs} < b < e.$$

This solutions can be expressed by a trigonometric and hyperbolic functions as follows:

$$W(\xi) = C_1 cosh(\alpha_1 \xi) + C_2 sinh(\alpha_1 \xi) + C_3 cos(\alpha_2 \xi) + C_4 sin(\alpha_2 \xi), \tag{10}$$

$$\Psi(\xi) = \overline{C_1} cosh(\alpha_1 \xi) + \overline{C_2} sinh(\alpha_1 \xi) + \overline{C_3} cos(\alpha_2 \xi) + \overline{C_4} sin(\alpha_2 \xi). \tag{11}$$

where:

$$\alpha_1 = \sqrt{\frac{-\eta_1 + \sqrt{\eta_1^2 - 4\eta_2}}{2}} \quad \text{and} \quad \alpha_2 = \sqrt{\frac{\eta_1 + \sqrt{\eta_1^2 - 4\eta_2}}{2}} \tag{12}$$

In the Eq. (10) and Eq. (11), C_i and $\overline{C_i}$ are dependents constants and $W(\xi)$ and $\Psi(\xi)$ are the modal shapes.

The second case gives:

$$\eta_1 > 0, \text{ which leads to: } b > \frac{1}{rs} \text{ and } b > e \text{ or } b < \frac{1}{rs} \text{ and } b < e$$

As result, the solution is expressed only in trigonometric functions:

$$W(\xi) = C_1^{'} cos(\alpha_3 \xi) + C_2^{'} sin(\alpha_3 \xi) + C_3^{'} cos(\alpha_2 \xi) + C_4^{'} sin(\alpha_2 \xi) \tag{13}$$

$$\Psi(\xi) = \overline{C_1^{'}} cos(\alpha_1 \xi) + \overline{C_2^{'}} sin(\alpha_1 \xi) + \overline{C_3^{'}} cos(\alpha_2 \xi) + \overline{C_4^{'}} sin(\alpha_2 \xi) \tag{14}$$

where

$$\alpha_3 = \sqrt{\frac{-\eta_1 - \sqrt{\eta_1^2 - 4\eta_2}}{2}}, \tag{15}$$

and $C_i^{'}$ and $\overline{C_i^{'}}$ are constants.

In the literature, some researchers have adopted a terminology to separate these pairs in two distinct spectra: the 'first spectrum' for $\eta_1 < 0$ and the 'second spectrum' for $\eta_1 > 0$.

3. FINITE ELEMENT METHOD

Consider a uniform axial-loaded Timoshenko beam element on Pasternak Foundation with length $2a$ as showed in Fig. 2. The beam element consists of two nodes and each node has two degrees of freedom: w, the total displacement and ψ, the total slope.

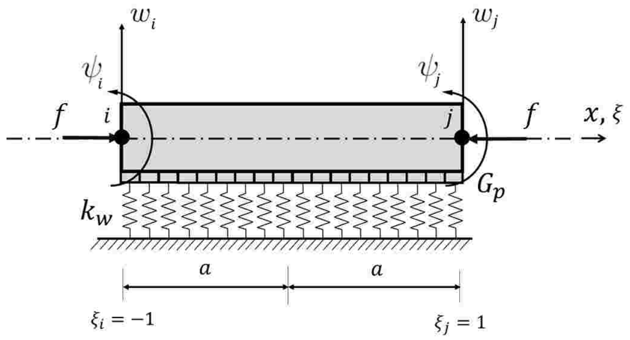

Figure 2: Beam element of a Timoshenko beam resting on a Pasternak foundation subjected to an axial load.

Applying the Rayleigh-Ritz method using polynomial functions approximations and applying the boundary conditions at the points $\xi_i = -1$ and $\xi_j = 1$, the displacement and slope matrices can be written as:

$$w = [\mathbf{N}(\xi)]_w \{\mathbf{v}\}_e \; and \; \psi = [\mathbf{N}(\xi)]_\psi \{\mathbf{v}\}_e \tag{16}$$

where $[\mathbf{N}(\xi)]_w$ and $[\mathbf{N}(\xi)]_\psi$ are the shape functions for displacement and slope, respectively, and $\{\mathbf{v}\}_e$ is the vector of nodal coordinates. The subscript e represents expressions for a single element. Therefore, the shape functions and the vector of nodal coordinates in Eq. (16), are expressed as:

$$[\mathbf{N}(\xi)]_w^T = \frac{1}{4(1+3\beta)} \begin{bmatrix} 2 - 3(1+2\beta)\xi + 6\beta + \xi^3 \\ a(1 - \xi - (1+3\beta)\xi^2 + 3\beta + \xi^3) \\ 2 + 3(1+2\beta)\xi + 6\beta - \xi^3 \\ a(-1 - \xi + (1+3\beta)\xi^2 - 3\beta + \xi^3) \end{bmatrix}, \tag{17}$$

$$[\mathbf{N}(\xi)]_\psi^T = \frac{1}{4(1+3\beta)} \begin{bmatrix} (-3 + 3\xi^2)/a \\ -1 - (2+6\beta)\xi + 6\beta + 3\xi^3 \\ (3 - 3\xi^2)/a \\ -1 + (2+6\beta)\xi + 6\beta + 3\xi^3 \end{bmatrix}, \tag{18}$$

$$\{\mathbf{v}\}_e^T = [w_i \quad \psi_i \quad w_j \quad \psi_j], \tag{19}$$

where $\beta = EI/(\kappa G A a^2)$.

Thus, considering the foundation, the axial load and the displacement and slope form matrices in the Eq. (16), the potential energy for an element $2a$ is given by:

$$\mathbf{U_e} = \frac{1}{2}\{\mathbf{v}\}_e^T \left[\frac{EI}{a}\int_{-1}^{1} [\mathbf{N}'(\xi)]_\psi^T [\mathbf{N}'(\xi)]_\psi d\xi + \frac{\kappa G A}{a}\int_{-1}^{1} \Big[[\mathbf{N}'(\xi)]_w - a[\mathbf{N}(\xi)]_\psi\Big]^T \Big[[\mathbf{N}'(\xi)]_w - a[\mathbf{N}(\xi)]_\psi\Big] d\xi + \right.$$
$$\left. k_w a \int_{-1}^{1} [\mathbf{N}(\xi)]_w^T [\mathbf{N}(\xi)]_w d\xi + \frac{G_p}{a}\int_{-1}^{1} [\mathbf{N}'(\xi)]_w^T [\mathbf{N}'(\xi)]_w d\xi + \frac{f}{a}\int_{-1}^{1} [\mathbf{N}'(\xi)]_w^T [\mathbf{N}'(\xi)]_w d\xi \right] \{\mathbf{v}\}_e, \tag{20}$$

where $[\mathbf{N}'(\xi)] = \partial[\mathbf{N}(\xi)]/\partial\xi$. Therefore, the kinetic energy is given by:

$$\mathbf{T_e} = \frac{1}{2}\{\dot{\mathbf{v}}\}_e^T \left[\rho A a \int_{-1}^{1} [\mathbf{N}(\xi)]_w^T [\mathbf{N}(\xi)]_w d\xi + \rho I a \int_{-1}^{1} [\mathbf{N}(\xi)]_\psi^T [\mathbf{N}(\xi)]_\psi d\xi \right] \{\dot{\mathbf{v}}\}_e, \tag{21}$$

where $\{\dot{\mathbf{v}}\} = \partial\{\mathbf{v}\}/\partial t$. Therefore, the elementary stiffness and mass matrices, $[k_e]$ and $[m_e]$, respectively are given by:

$$[\mathbf{k_e}] = \frac{EI}{a}\int_{-1}^{1} [\mathbf{N}'(\xi)]_\psi^T [\mathbf{N}'(\xi)]_\psi d\xi + \frac{\kappa G A}{a}\int_{-1}^{1} \Big[[\mathbf{N}'(\xi)]_w - a[\mathbf{N}(\xi)]_\psi\Big]^T \Big[[\mathbf{N}'(\xi)]_w - a[\mathbf{N}(\xi)]_\psi\Big] d\xi +$$
$$k_w a \int_{-1}^{1} [\mathbf{N}(\xi)]_w^T [\mathbf{N}(\xi)]_w d\xi + \frac{G_p}{a}\int_{-1}^{1} [\mathbf{N}'(\xi)]_w^T [\mathbf{N}'(\xi)]_w d\xi + \frac{f}{a}\int_{-1}^{1} [\mathbf{N}'(\xi)]_w^T [\mathbf{N}'(\xi)]_w d\xi, \tag{22}$$

$$[\mathbf{m_e}] = \rho A a \int_{-1}^{1} [\mathbf{N}(\xi)]_w^T [\mathbf{N}(\xi)]_w d\xi + \rho I a \int_{-1}^{1} [\mathbf{N}(\xi)]_\psi^T [\mathbf{N}(\xi)]_\psi d\xi. \tag{23}$$

Finally, the natural frequencies of the beam are founded applying the Lagrange equation and solving the eigenvalues and eigenvectors problems.

$$\left([\mathbf{K}] - [\mathbf{M}]\omega_n^2\right)\{\mathbf{v}\} = 0, \tag{24}$$

where $[\mathbf{K}]$, $[\mathbf{M}]$ and $\{\mathbf{v}\}$ are the global inertia matrix, global stiffness matrix and the global vector of nodal coordinates, that are derived of the elementary stiffness matrix, elementary inertia matrix and vector of nodal coordinates Eq. (22), (23) and (19), with variation dependent on the number of elements.

4. NUMERICAL RESULTS AND DISCUSSION

In this section, a numerical example presented by Abohadima *et al.* (2015) is analyzed. A uniform beam of uniform cross-section area such that $\nu = 1/4$, $E/G = 2.5$, the shear factor given by $\kappa = 2/3$, $\rho = 2500kg/m^3$, $I = 0,05^2 m^4$ and $A = 1m^2$. The first three frequency parameters obtained for different values of the linear elastic spring parameter (e), the shear layer parameter (p) and the axial load parameter (n), cases are presented on Tab. 1. Finite elements (discretization with 30 and 50 elements) results are in agreement with literature (Abohadima *et al.*, 2015).

Table 1: First three frequencies parameters for a hinged-hinged beam.

Parameters			b_1			b_2			b_3		
n	e	p	30 ele.	50 ele.	Exact$^{(1)}$	30 ele.	50 ele.	Exact$^{(1)}$	30 ele.	50 ele.	Exact$^{(1)}$
0	0	0	2.866	2.866	2.866	4.925	4.923	4.922	6.454	6.449	6.446
0.6	0	0	1.862	1.862	1.863	4.387	4.384	4.384	5.932	5.926	5.923
0.6	$0.6\pi^4$	0	2.866	2.866	2.866	4.540	4.538	4.538	5.997	5.991	5.988
0.6	$0.6\pi^4$	π^2	3.555	3.555	3.555	5.296	5.294	5.294	6.785	6.779	6.777

$^{(1)}$ (Abohadima *et al.*, 2015)

In Table 1 it is verified that the presence of axial load decreases the frequency parameters b_i. This behavior occurs because the transverse component of the axial load causes an increase in the shear force. In addition, the influence of the foundation on the free vibration of an beam subjected to axial load is observed. The addition of the first foundation parameter causes a significant increase in the frequency parameter in the first mode, although it becomes less significant for higher modes. The results for the second foundation parameter (Pasternak Foundation) show a notable increase for all vibration modes presented, being higher than those presented by the first foundation parameter. Also, variations in the frequency parameter decrease as the mode number increases. For higher modes, as expected, the difference between the FEM and the analytic solutions decreases as the number of elements increases. Finally, the biggest error was 0.152%, perceived in the third frequency using 30 elements.

4.1 Hinged-Hinged beam

4.1.1 Without foundation and without axial load

The first example for evaluating the frequency curves consists of a hinged-hinged Timoshenko beam, considering that there is no foundation and no axial load action, that is, $n = 0$, $e = 0$ and $p = 0$. Frequency curves were plotted for different Rayleigh factor r values to assess the influence of beam slenderness in the frequencies. The results for this case can be seen in Fig. 3. All frequency curve results are made using 70 elements.

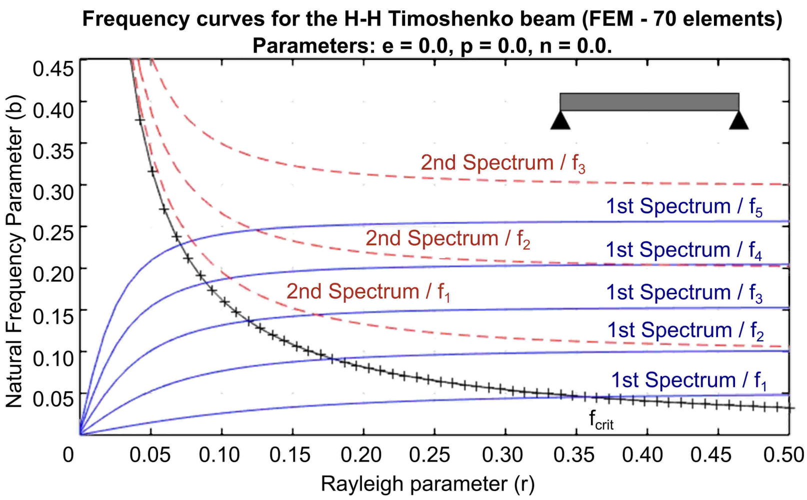

Figure 3: Frequency curves for the hinged-hinged Timoshenko beam.

In Fig. 3, solid-lines represent the first spectrum, the dashed lines represent the second spectrum and the shear mode is the curve marked by '+'. Note that the shear mode frequency is not a boundary between the two frequency spectrums. This also can be noticed for r parameter equal to 0.25, for example, whose the shear mode curve is contained between two curves belonging to first spectrum. However, for each r parameter, the natural frequencies from second spectrum are above shear mode frequencies. Therefore, as noticed by Vasconcelos *et al.* (2016), this spectrum appears only for higher frequencies than shear mode. Additionally, as r factor increases, the higher the frequencies of the first spectrum. On the other hand, the second spectrum frequencies become lower.

4.1.2 Winkler foundation without axial load

Considering now a hinged-hinged Timoshenko beam resting on a one-parameter elastic foundation without axial load action, such that: $n = 0$, $e = 0.6\pi^4$ and $p = 0$. Figure 4 shows the variation of the frequencies of the two spectra as a function of the r parameter.

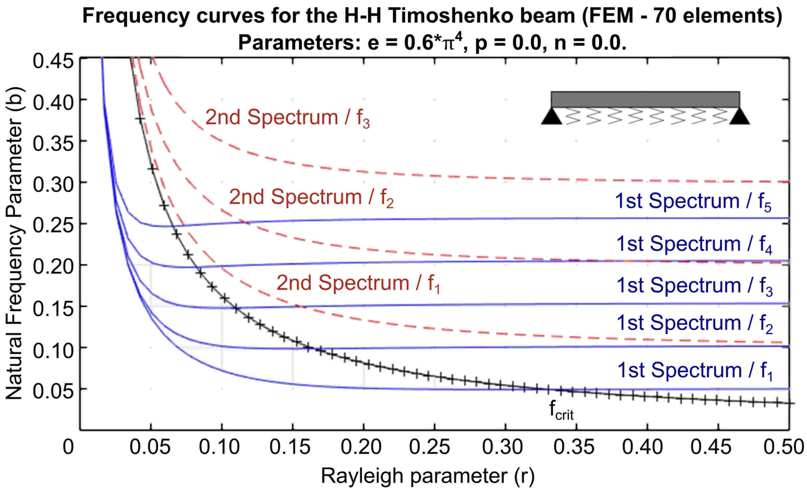

Figure 4: Frequency curves for the hinged-hinged Timoshenko beam on Winkler foundation.

From the graph in Fig. 4 is noted that, the inclusion of the first parameter has no effect on the frequencies of the second spectrum and the frequency of the shear mode. On the another hand, the first spectrum frequencies decrease as the r parameter increases, up to approximately $r = 0.15$. From there, the frequency values tend to remain constant. This occurs because thin beams are more influenced by foundation, creating a singularity.

4.1.3 Two-parameter foundation without axial load

Considering a Timoshenko beam on two parameter elastic foundation without axial load, in this case: $n = 0$, $e = 0.6\pi^4$ and $p = \pi^2$. The results of frequency curves for this case are shown in the Fig. 5.

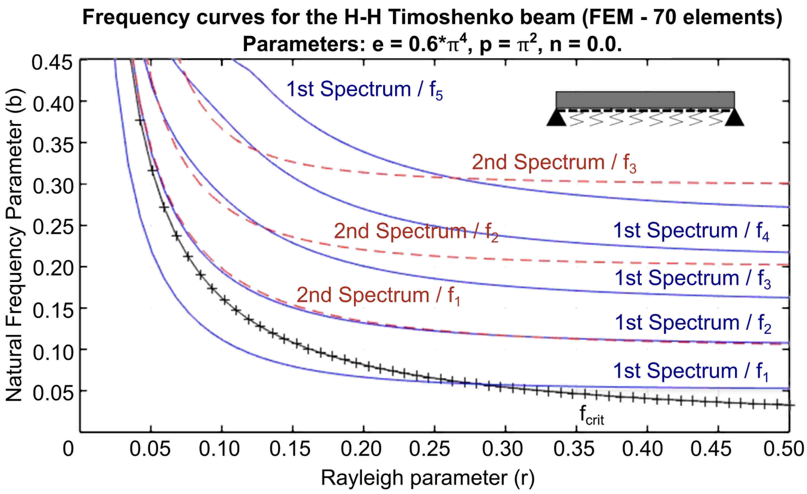

Figure 5: Frequency curves for the hinged-hinged Timoshenko beam on Pasternak foundation.

The results of Fig. 5 show that when the first and second foundation parameters are considered, all frequencies decrease, similarly to the previous case (section 4.1.3). Moreover, the addition of the second parameter promoted an increase in the frequencies of the first spectrum, which behavior was more pronounced for the second mode onwards. In this case, the influence of the parameter is noticed both in thin beams and in thicker beams, since the frequency curve tends to become more constant when the r parameter approaches 0.5. It is also interesting to note that the frequencies of the second spectrum were affected, with the second and third modes having more pronounced increases.

4.1.4 Two-parameter foundation with axial load

Finally, considering a hinged-hinged timoshenko axial loaded beam resting on a two-parameter elastic foundations such that $n = 0.2$, $e = 0.6\pi^4$ and $p = \pi^2$. Figure 6 shows the natural frequency parameter curves for various values of the r parameter.

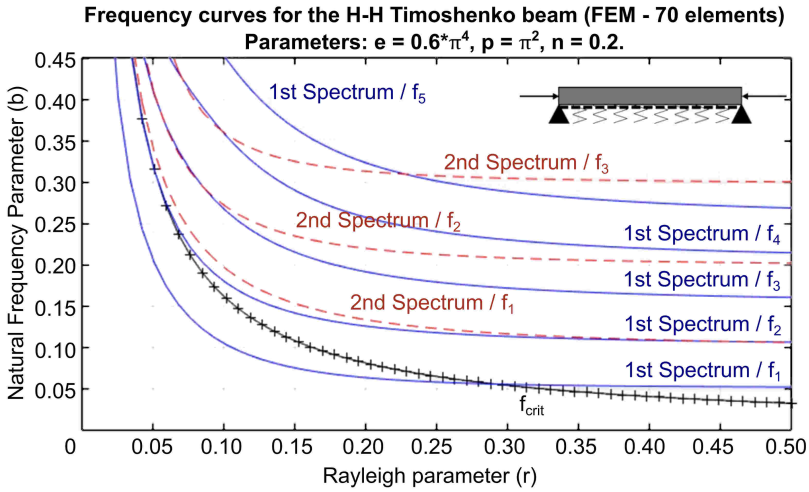

Figure 6: Frequency curves for the hinged-hinged Timoshenko beam-column on Pasternak foundation.

Figure 6 shows that the curves are very similar to the frequency curves shown in Fig. 5. This similarity of the curves is due to the fact that the influence of the second foundation parameter in increasing the system frequencies is greater than the influence of the load parameter in decreasing the system frequencies. However, it is possible to notice a small decline in the frequency curves caused by the axial load. This behavior is in agreement with the observations made in the numerical example in Table 1.

4.2 Clamped-Clamped beam

4.2.1 Without foundation and without axial load

As a second comparison, the same graphs are generated for the clamped-clamped beam case and present the following results. Figure 7 shows the results of the frequency curves for the clamped-clamped beam without axial load and without foundation such that $n = 0$, $e = 0$ and $p = 0$.

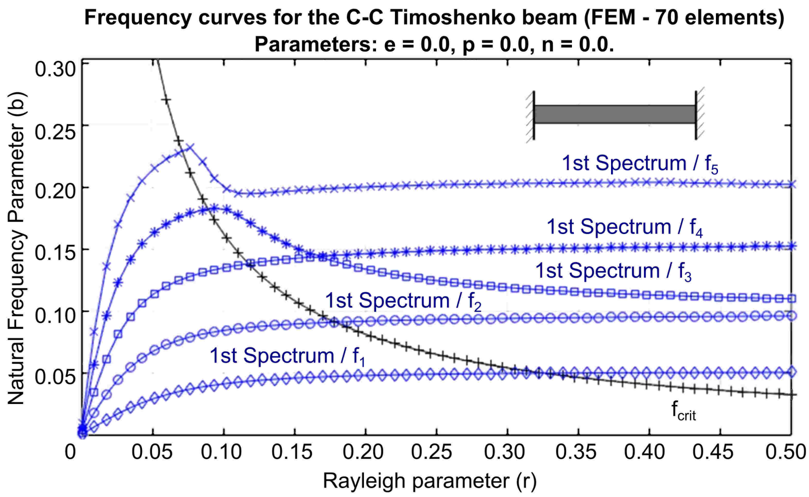

Figure 7: Frequency curves for the clamped-clamped Timoshenko beam.

From the graph in Fig. 7, it can be seen that the the behavior of the natural frequency parameter and the critical frequency (f_{crit}) are similar to that observed in the same case for a hinged-hinged beam (section 4.1.1). Moreover, it is noticeable that for the clamped-clamped beam there is not second spectrum phenomena as observed in hinged-hinged case. At another point, in this case, the crossing of modes between the 3rd and 4th vibration mode is verified, which can be identified in the graph but was not addressed by the algorithm.

4.2.2 Winkler foundation without axial load

Considering a clamped-clamped Timoshenko beam resting on a Winkler foundation and without axial load action, that is, $n = 0$, $e = 0.6\pi^4$ and $p = 0$. The results for frequency curves for this case can be seen in Fig. 8.

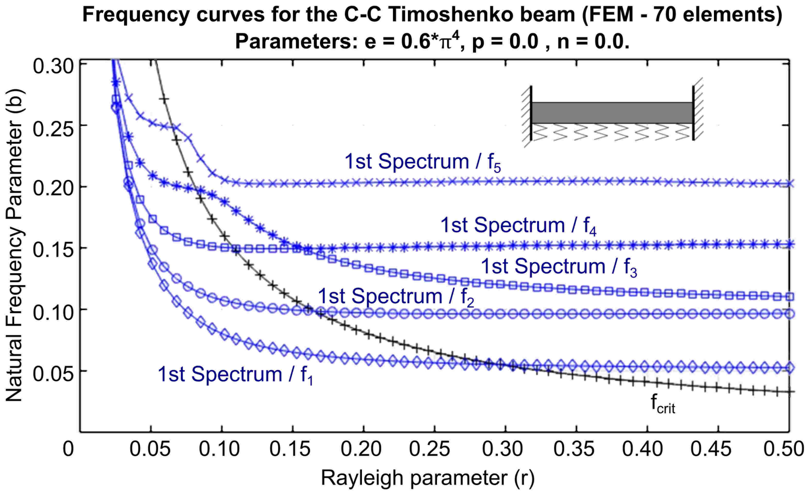

Figure 8: Frequency curves for the clamped-clamped Timoshenko beam on Winkler foundation.

In contrast to the previous case, the first foundation parameter causes all frequencies to decrease mainly for r parameter lower than 0.1, creating a singularity in $r = 0$. In this case, similarly to that observed in the same condition for the hinged-hinged beam (section 4.1.2), for thin beams (with $r < 0.1$) the presence of foundation increase the natural frequency by increasing of system inertia and for thick beams that increase is not enough to change the system inertia.

4.2.3 Two-parameter foundation without axial load

The foundation parameter contribution withou the compressive axial load contribution is shown in Fig 9. In this case, the foundation parameters are: $e = 0.6\pi^4$ and $p = \pi^2$, and the axial load parameter is: $n = 0$.

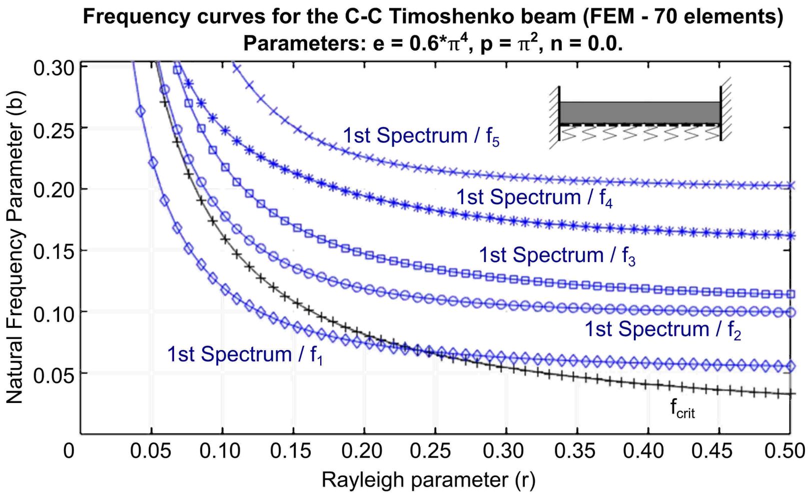

Figure 9: Frequency curves for the clamped-clamped Timoshenko beam on Pasternak foundation.

From the graph in Fig. 9, it can be seen that the addition of the second parameter promotes a much more pronounced increase in the frequency curves than with the addition of the Winkler parameter alone, a behavior that has more effect from the second mode onwards. These results corroborate what has already been observed in the numerical examples in Tab. 1 and the hinged-hinged beam for the same foundation condition and axial load.

4.2.4 Two-parameter foundation with axial load

Finally, considering a clamped-clamped Timoshenko beam axial-loaded resting in a two-parameter foundation, such that: $e = 0.6\pi^4$, $p = \pi^2$, and $n = 0.2$. Figure 10 presents the results of the frequency curves for this case.

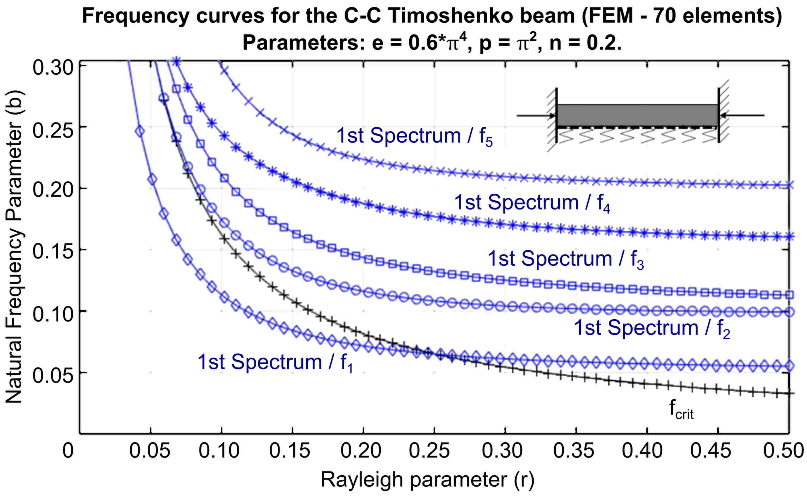

Figure 10: Frequency curves for the clamped-clamped Timoshenko beam-column on Pasternak foundation.

Looking at the graph in Fig. 10, we have that the contribution of the foundation parameters and the axial load are practically the same presented in the articulated beam for this case, that is, the foundation increases the frequency parameter due to the increase in the stiffness of the system and the parameter of axial load causes a decrease in the frequency parameter due to the shear effect induced by the axial load.,

5. CONCLUSION

In this paper, the full development and analysis of an axial-loaded Timoshenko beam on two parameter foundation formulation was presented for several numerical examples of the effect of the Winkler and Pasternak parameter and the axial load parameter for the transversely vibrating uniform beam for boundary conditions hinged-hinged and clamped-clamped. A finite element was sucessfully developed from the energy conservation based upon Hamilton's principle. A cubic and quadratic Lagrangian polynomial approximated is used for total deflection and slope. Numerical examples are presented for two boundary conditions in order to study beam behavior around critical frequency. The investigation between the frequency parameter and the Rayleigh parameter verified the existence of a second spectrum in the hinged-hinged boundary condition and its absence in the clamped-clamped beam, as expected. Finally, the finite element results were in agreement with the results of other researchers.

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